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Volume of n-dimensional ellipsoid

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Abstract. In this article the volume of the n-dimensional ellipsoid is derived using the method, step by step process of integration. Recurrence relations are developed to find the volume and surface area of n-dimensional sphere. The relation between the volume and surface area of n-dimensional sphere is given. The asymptotic behavior of the volume and surface area of the unit sphere is also discussed.
Key words: n-dimensional ellipsoid, volume, surface area, asymptotic behavior, Euclidean space.

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1 Introduction

Let R^n be the n-dimensional Euclidean space. The equation to the n-dimensional ellipsoid is given by $\sum_{i=1}^{n} \frac{x_i^2}{a_i^2} = 1$ where a_i denotes the length of the semi axes of the ellipsoid. The ellipsoid axes are aligned with the Cartesian coordinate axes of the n-dimensional Euclidean space. A method to determine the volume of of the ellipsoid is being proposed in this article. This method is based on a step by step process in which the dimension is reduced by one in each step.

2 Evaluation of volume

Theorem 2.1. *The volume of the* n*-dimensional ellipsoid is given by*

$$V_n = \frac{2}{n}\frac{\pi^{n/2}}{\Gamma(n/2)}(a_1, a_2, a_3 \dots a_n)$$

where $a_1, a_2, a_3 \dots a_n$*, denote the length of semi axes of the* n*-dimensional ellipsoid.*

Proof. Let V_n be the volume of the n-dimensional ellipsoid $\sum_{i=1}^{n} \frac{x_i^2}{a_i^2} = 1$.
$V_1$is the volume for $\frac{x_1^2}{a_1^2} = 1$, denoting the length of the segment from $-a_1$ to $+a_1$ which is equal to $2a_1$.
V_2 is the area of the ellipse $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1$.

$$\begin{aligned} V_2 &= \pi a_1 a_2 \\ &= a_1 a_2 2\beta\left(\tfrac{3}{2}, \tfrac{1}{2}\right) \end{aligned}$$

$V_3$is the volume of the three dimensional ellipsoid $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$. We can write it as

$$\frac{x_1^2}{\left(\frac{a_1}{a_3}\right)^2 \left(a_3^2 - x_3^2\right)} + \frac{x_2^2}{\left(\frac{a_2}{a_3}\right)^2 \left(a_3^2 - x_3^2\right)} = 1.$$

It is an ellipse with varying semi axes $\frac{a_1}{a_3}\sqrt{a_3^2 - x_3^2}$ and $\frac{a_2}{a_3}\sqrt{a_3^2 - x_3^2}$ ranging from $-a_3$ to a_3. The required volume is the collection of elliptic plates arranged one above the other. Hence

$$\begin{aligned} V_3 &= \pi\frac{a_1}{a_3}\frac{a_2}{a_3}\int_{-a_3}^{a_3} \left(a_3^2 - x_3^2\right) dx_3 \\ &= \pi\frac{a_1}{a_3}\frac{a_2}{a_3} 2a_3^3 \int_0^{\frac{\pi}{2}} \cos^3\theta d\theta, \quad \text{under the substitution } x_3 = a_3 \sin\theta \\ &= a_1 a_2 a_3 2\beta\left(\frac{3}{2}, \frac{1}{2}\right)\beta\left(2, \frac{1}{2}\right) \end{aligned}$$

V_4 is the volume of the fourth dimensional ellipsoid

$$\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}+\frac{x_4^2}{a_4^2}=1$$

which can be written as

$$\frac{x_1^2}{\left(\frac{a_1}{a_4}\right)^2\left(a_4^2-x_4^2\right)}+\frac{x_2^2}{\left(\frac{a_2}{a_4}\right)^2\left(a_4^2-x_4^2\right)}+\frac{x_3^2}{\left(\frac{a_3}{a_4}\right)^2\left(a_4^2-x_4^2\right)}=1$$

It's volume is

$$\begin{aligned} V_4 &= \frac{a_1}{a_4}\frac{a_2}{a_4}\frac{a_3}{a_4}2\beta\left(\frac{3}{2},\frac{1}{2}\right)\beta\left(2,\frac{1}{2}\right)\int\limits_{-a_4}^{a_4}\left(a_4^2-x_4^2\right)^{\frac{3}{2}}dx_4 \\ &= a_1a_2a_3a_4 2\beta\left(\frac{3}{2},\frac{1}{2}\right)\beta\left(2,\frac{1}{2}\right)\beta\left(\frac{5}{2},\frac{1}{2}\right) \end{aligned}$$

Generalizing this

$$\begin{aligned} V_n &= a_1a_2a_3\cdots a_n 2\beta\left(\frac{3}{2},\frac{1}{2}\right)\beta\left(2,\frac{1}{2}\right)\beta\left(\frac{5}{2},\frac{1}{2}\right)\cdots \quad \cdots\beta\left(\frac{n}{2},\frac{1}{2}\right)\beta\left(\frac{n+1}{2},\frac{1}{2}\right) \\ &= (a_1a_2a_3\cdots a_n)\frac{2}{n}\frac{\pi^{\frac{n}{2}}}{\Gamma(n/2)} \quad \text{(on simplification)} \end{aligned}$$

∎

3 Special cases

The volume of the n-dimensional sphere is

$$V_n=\frac{2}{n}\frac{\pi^{\frac{n}{2}}}{\Gamma(n/2)}a^n, \quad \text{where } a \text{ is the radius of the sphere.} \tag{1}$$

The recurrence formula for the volume of the n-dimensional sphere is given by

$$V_{n+2}=\frac{2\pi a^2}{n+2}V_n \quad \text{with} \quad V_0=1, V_1=2a$$

The concept of volume is meaningful for $n\geq 3$, for $n=2$ it coincides with the common concept of area of a circle and for $n=1$ with the length of a one dimensional interval and for $n = 0$, corresponds to a zero-dimensional sphere of volume $= 1$. On differentiating the volume with respect to the radius we get the surface area. Hence the surface area is

$$S_n = \frac{dV_n}{da} = \frac{2a^{n-1}\pi^{\frac{n}{2}}}{\Gamma\left(n/2\right)}. \tag{2}$$

The recurrence formula for the surface area of the sphere is given by

$$S_{n+2} = \frac{2\pi a^2}{n} S_n,\ n \neq 0 \quad \text{with} \quad S_0 = 0 \quad \text{and} \quad S_1 = 2.$$

From (1) and (2), the volume and the surface area of n-dimensional sphere are related by

$$V_n = \frac{a}{n} S_n$$

The volume and surface area of the unit sphere is given by

$$V_n = \frac{2}{n}\frac{\pi^{\frac{n}{2}}}{\Gamma(n/2)} \quad \text{and} \quad S_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(n/2)}$$

4 Asymptotic behavior

Using the recurrence relations the volume and surface area of the unit sphere for different dimensions are given in Table 1 and are illustrated in Figs. 1 and 2.

It is clear that the volume reaches a maximum for $n = 5$ and the surface area reaches a maximum for $n = 7$ and decays to zero as the dimension increases.

5 Conclusion

From the general formula we can find the volume and surface area for any dimension. The volume and surface area for different dimensions are graphically presented. As the dimension is large both volume and surface area asymptotically tends to zero, is an interesting phenomenon.

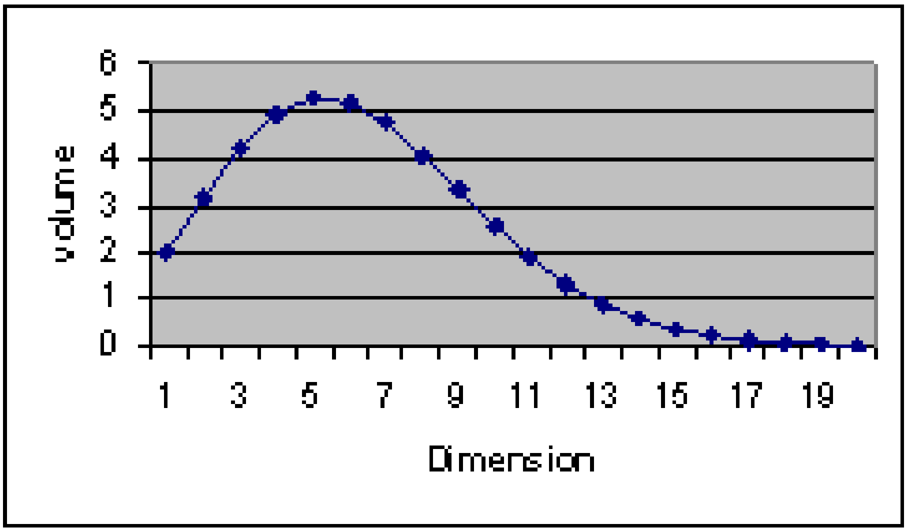

Figure 1: Asymptotic behaviour of volume.

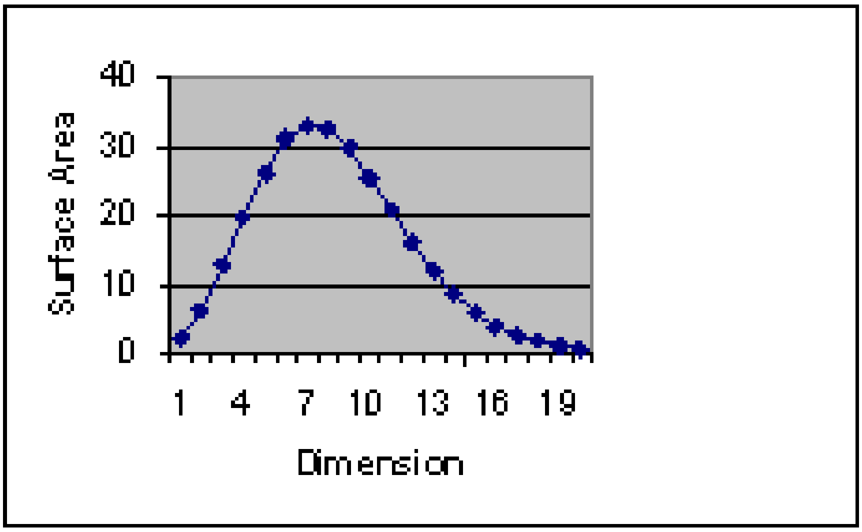

Figure 2: Asymptotic behaviour of surface area.

Table 1:

Dimension	Volume	Numerical values	Surface area	Numerical values
1	2	2.000000	2	02.00000
2	π	3.141592	2π	06.28319
3	$4\pi/3$	4.188790	4π	12.56637
4	$\pi^2/2$	4.934802	$2\pi^2$	19.73921
5	$8\pi^2/15$	5.263789	$8\pi^2/3$	26.31895
6	$\pi^3/6$	5.167712	π^3	31.00628
7	$16\pi^3/105$	4.724765	$16\pi^3/15$	33.07336
8	$\pi^4/24$	4.058712	$\pi^4/3$	32.46969
9	$32\pi^4/945$	3.298508	$32\pi^4/105$	29.68658
10	$\pi^5/120$	2.550164	$\pi^5/12$	25.50164
11	$64\pi^5/10395$	1.884104	$64\pi^5/945$	20.72514
12	$\pi^6/720$	1.335263	$\pi^6/60$	16.02315
13	$128\pi^6/135135$	0.910628	$128\pi^6/10395$	11.83817
14	$\pi^7/5400$	0.599264	$\pi^7/360$	08.38970
15	$256\pi^7/2027025$	0.381443	$256\pi^7/135135$	05.72165
16	$\pi^8/40320$	0.23533	$\pi^8/2520$	03.76529
17	$512\pi^8/34459425$	0.14098	$512\pi^8/2027025$	02.39668
18	$\pi^9/362880$	0.08215	$\pi^9/20160$	01.47863
19	$1024\pi^9/654729075$	0.04662	$1024\pi^9/34459425$	00.88581
20	$\pi^{10}/3628800$	0.02581	$\pi^{10}/181440$	00.51614

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